Topological terms in sigma models on homogeneous spaces

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Outline of talk

1. Overview & motivation
2. Why singular homology?
3. Classification in two parts: Aharonov-Bohm (AB) and Wess-Zumino (WZ)
4. Application to Composite Higgs Models

Overview & motivation

Overview

Sigma model on a homogeneous space = a QFT of maps

$$\phi : \Sigma^p \to G/H,$$

with dynamics described by a G-invariant action phase

$$e^{2\pi i S[\phi]}$$

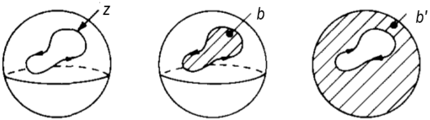
For topological terms, we will replace maps with p-cycles, and define action phase by integrating (locally-defined) differential forms on cycles. Two types:

1. Aharonov-Bohm (AB) terms - integrate p-form A, $dA = 0$
2. Wess-Zumino (WZ) terms - integrate p-forms A_α, $dA_\alpha = \omega$ is a closed, integral, globally-defined $(p+1)$-form

Motivation I: hep-th

Recall Witten's construction of the WZW term:[1]

- Assumes Σ^p homeomorphic to S^p
- If $\pi_p(G/H) = 0$, any such Σ^p is boundary of a $(p+1)$-ball B in G/H
- Can then write topological action as $\int_B \omega$



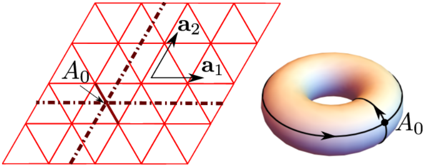
E.g. in chiral lagrangian, $\pi_4(SU(3)) = 0$

[1] Witten, 1983.

Motivation I: hep-th

Two limitations of this approach:

1. Want to define WZW on worldvolumes of arbitrary topology (not just S^p):
    - $S^{p-1} \times S^1$: chiral lagrangian in the background of a skyrmion
    - T^p: periodic BCs *e.g.* in condensed matter



    - In cosmology - what is topology of Universe?

Easy resolution: switch from homotopy to homology.

Motivation I: hep-th

Two limitations of this approach:

1. Want to define WZW on worldvolumes of arbitrary topology
   Easy resolution: switch from homotopy to homology

2. Want to define WZW on *all* maps/cycles (not just those homotopic to id). *E.g.*:
   - QM on T^n (non-trivial 1-cycles)
   - Composite Higgs where $H_4(G/H) \neq 0$, e.g. $G/H = SO(5)/SO(4)$, $SO(6)/SO(4)$ (non-trivial 4-cycles)

   Resolution: integrate locally-defined forms. But G-invariance will be more subtle...

Motivation II: hep-ph

Composite Higgs solutions to EW hierarchy problem:
H as pNGB of symmetry breaking $G \to H \supset SU(2)_L \times SU(2)_R$ in some strong sector

naturally light, *c.f.* pions of low-energy QCD

CH lives on compact space G/H - generic possibility of topological terms

Motivation II: hep-ph

Topological terms give insight into the UV completion of IR sigma model, by anomaly matching.

Example: chiral lagrangian

Gauged WZ term gives LO contribution to $\pi \to \gamma\gamma$, and $n_{WZ} = N_c$ in underlying $SU(N_c)$ gauge theory by anomaly matching - *measure* $N_c = 3$ in QCD!

They can also affect the IR pheno. *c.f.* WZW term in chiral lagrangian gives leading contribution to $KK \to \pi\pi\pi$

And lots more...

- $p = 1$, Landau levels on G/H
- $p = 2$, strings/ CFTs
- $p \geq 3$, Goldstone bosons - *everywhere*...
  1. particle physics (pions, CH)
  2. condensed matter (fluids, superfluids)
  3. cosmology (galileons)

Formalism: why singular homology?

Defining topological terms

Assume Σ^p smooth, connected, orientable. Equip Σ^p with an orientation $\rightarrow$ can integrate differential forms

'Topological terms' require no further structure on Σ^p (no metric), and will be invariant under $\mathcal{O}$, group of orientation-preserving diffeos of Σ^p

Why chains/ cycles?

We will *not* integrate forms on *manifold* Σ^p, but on *chains/ cycles*. Why?

Want to make frequent use of de Rham's theorems:

A p-form has vanishing integral over every

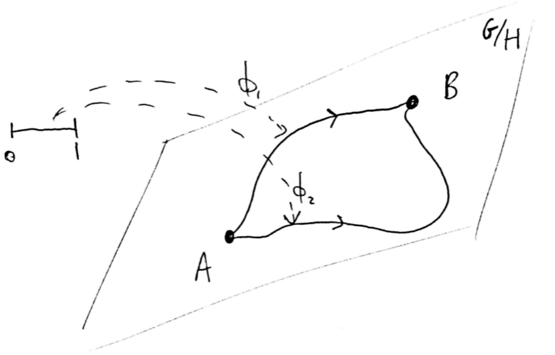
$$\left\{ \begin{array}{l} p\text{-chain} \\ p\text{-cycle} \\ p\text{-boundary} \end{array} \right\} \text{iff. it} \left\{ \begin{array}{l} \text{vanishes.} \\ \text{is exact.} \\ \text{is closed.} \end{array} \right\} \quad (1)$$

Requires integration on chains/ cycles

From maps to cycles

One final assumption about worldvolume: Σ^p is *closed*.
Justification:

In $p = 1$ (QM), only *relative* paths matter



In QFT, finite action requires fields die off "at infinity"

From maps to cycles

Upshot of these assumptions is that

$$H_p(\Sigma^p, \mathbb{Z}) = \mathbb{Z},$$

generated by the 'fundamental class' $[\Sigma^p]$ (a p-cycle). Invariant under $\mathcal{O}$.

Push-forward any cycle in $[\Sigma^p]$ using ϕ to define a cycle z in G/H

Integrate p-form on G/H on z to obtain an action. Well-defined on $[\Sigma^p]$, and therefore $\mathcal{O}$-invariant

We shall assume action phase defined on all p-cycles in G/H (want to use de Rham's theorems)

We shall allow the p-forms on G/H that we integrate to be only locally-defined

The target space G/H

G is a connected Lie group, otherwise arbitrary, and H is a Lie subgroup of G

The classification: AB and WZ terms

AB terms

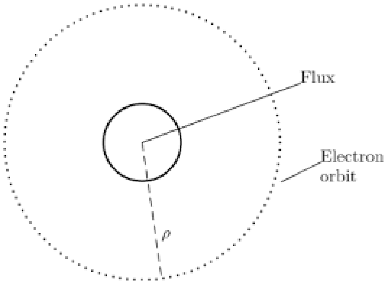
Integrate closed p-forms A on p-cycle z:

$$S[z] = \int_z A \tag{2}$$

- Locally a total derivative in lagrangian $\rightarrow$ doesn't affect classical EOMs
- doesn't affect perturbation theory
- only non-zero if there are non-trivial cycles in G/H, i.e. $H_p(G/H, \mathbb{Z}) \neq 0$
- Only depends on de Rham cohomology class of A

Example: QM on circle

- $S[z] = \int_z A = \int_z \frac{b}{2\pi} d\theta = bW[\phi]$, $W \in \mathbb{Z}$ is winding number
- $e^{2\pi ibW}$ is Aharonov-Bohm phase acquired by wavefunction



- $e^{2\pi ibW}$ invariant (on *all* cycles) upon $b \to b + n$. Hence quotient by $b \in \mathbb{Z} \implies b \in \mathbb{R}/\mathbb{Z} \simeq U(1)$

Classifying AB terms

Automatically G-invariant:

$$\delta_X S[z] = \int_z L_X A = \int_z d\iota_X A = \int_{\partial z} \iota_X A = \int_0 \iota_X A = 0, \qquad (3)$$

where L_X is Lie derivative, and X *any* vector field on G/H.

Classifying AB terms

Q: Are de Rham classes in 'one-to-one' inequivalent action phases?
A: No.

If $\int_z (A - B) \in \mathbb{Z}$ for any p-cycle z, then $\exp(2\pi i \int_z A)$ and $\exp(2\pi i \int_z B)$ will agree on all p-cycles (*c.f.* QM on S^1)

Thus space of physically inequivalent AB terms is thus

$$H^p_{dR}(G/H, \mathbb{R})/H^p(G/H, \mathbb{Z})_{\mathbb{R}}, \tag{4}$$

Another example: 2-d $\mathbb{C}P^N$ model

$H^2_{dR}(\mathbb{C}P^N, \mathbb{R}) = \mathbb{R}$, generated by Kähler form $A = \frac{i}{2} dz^i \wedge d\bar{z}^i$

In large N limit, theory is dual to QED in 2-d, with AB term $\to$ theta term[2]

[2] Schwinger, 1962

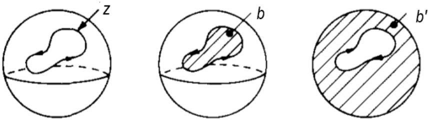
WZ terms

Integrate p-forms $\{A_\alpha\}$ which are not closed, and possibly only locally-defined on open sets $\{U_\alpha\}$

WZ terms

... but even if A_α only locally-defined, $\omega = dA_\alpha$ must be globally-defined since appears directly in classical EOMs

If $z = \partial b$, can define action á la Witten $S[z] = \int_b \omega$



G-invariance implies

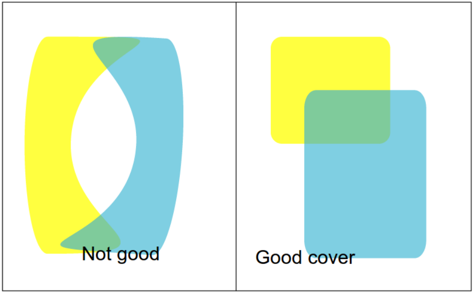
$$\delta_X S[z] = \int_b L_X \omega = 0 \quad \forall b \in C_{p+1} \implies L_X \omega = 0$$

But if $z \neq \partial b$ for any b, then must integrate local forms directly, and G-invariance obscured. Formulate using Čech cohomology,[3] which is about piecing together local information

[3] Alvarez, 1985

WZ terms: Čech cohomology basics

Choose a good cover $\mathcal{U} = \{U_\alpha\}$ on G/H, *i.e.* such that all finite intersections $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p} := U_{\alpha_0 \alpha_1 \ldots \alpha_p}$ are contractible.[4]



[4] Bott & Tu, 1982

WZ terms: Čech cohomology basics

Define a Čech p-cochain on $\mathcal{U}$ with values in $\mathcal{F}$, $\omega \in \check{C}^p(\mathcal{U}, \mathcal{F})$, by the set of values $\{\omega_{\alpha_0\alpha_1\ldots\alpha_p} \in \mathcal{F}(U_{\alpha_0\alpha_1\ldots\alpha_p})\}$

Define a Čech coboundary operator $\delta_p : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by its action on $\omega_{\alpha_0\alpha_1\ldots\alpha_p}$:

$$(\delta\omega)_{\alpha_0\alpha_1\ldots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i \omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}}, \tag{5}$$

s.t. $\delta_p \circ \delta_{p-1} = 0$

Define Čech cohomology groups in usual way,
$\check{H}(G/H, \mathcal{F}) = \ker \delta_p / \mathrm{im}\ \delta_{p-1}$.

WZ terms: locally-defined forms

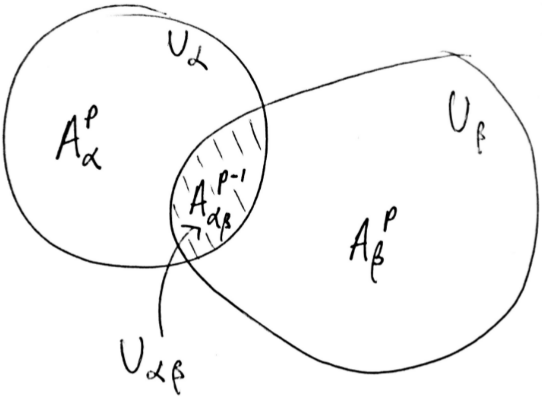
Choose a globally-defined, closed, $(p+1)$-form ω on G/H. Defines an element of $\check{C}^0(\mathcal{U}, \Lambda^{p+1})$ by restriction: $\omega_\alpha := \omega|_\alpha$.

Using Poincaré lemma, may construct an element $\{A^p_\alpha\} \in \check{C}^0(\mathcal{U}, \Lambda^p)$ via

$$dA^p_\alpha = \omega_\alpha, \text{ on } U_\alpha. \tag{6}$$

ω globally-defined $\implies d(A^p_\alpha - A^p_\beta) = 0$ on $U_{\alpha\beta}$. Using Poincaré lemma, may construct an element $\{A^{p-1}_{\alpha\beta}\} \in \check{C}^1(\mathcal{U}, \Lambda^{p-1})$ via

$$A^p_\alpha - A^p_\beta = dA^{p-1}_{\alpha\beta} \quad \text{on } U_{\alpha\beta} \iff \delta\{A^p_\alpha\} = \left\{dA^{p-1}_{\alpha\beta}\right\} \tag{7}$$



$$dA_\alpha^p = \omega_\alpha, \qquad A_\alpha^p - A_\beta^p = dA_{\alpha\beta}^{p-1}$$

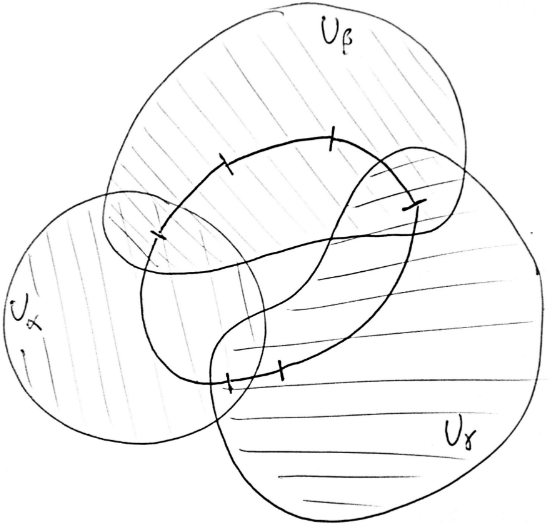
WZ terms: $\mathcal{U}$-small chains

To integrate p-forms $\{A^p_\alpha\}$, require chains contained within $\{U_\alpha\}$ ('$\mathcal{U}$-small chains')

Apply the *subdivision operator*,[5] Sd, as many times, n say, as is necessary

$z \mapsto \mathrm{Sd}^n z = \sum_\alpha c_{p,\alpha}$, where $\mathrm{Im}\ c_{p,\alpha} \subset U_\alpha$

[5] Vick, 1994


U_β
U_α
U_γ

Let's try to define an action phase with these objects

First attempt:

$$S[z] = \sum_{\alpha} \int_{c_{p,\alpha}} A^p_\alpha, \quad \text{where} \quad \mathrm{Sd}^n z = \sum_\alpha c_{p,\alpha} \quad ?$$

No. Ambiguous whenever a p-simplex σ is contained in intersection of two open sets, say $U_{\alpha\beta}$.

Fix: integrate $(p-1)$-form $A^{p-1}_{\alpha\beta}$ over $c_{(p-1),\alpha\beta}$, shared boundary of $c_{p,\alpha}$ and $c_{p,\beta}$

$$S[z] = \sum_{\alpha} \int_{c_{p,\alpha}} A^{p}_{\alpha} - \sum_{\alpha\beta} \int_{c_{(p-1),\alpha\beta}} A^{p-1}_{\alpha\beta}$$

Ambiguity vanishes because $A^{p}_{\alpha} - A^{p}_{\beta} - dA^{p-1}_{\alpha\beta} = 0$

But we are not done!

Ambiguity over which of the $\left\{A^{p-1}_{\alpha\beta}\right\}$ to integrate if a simplex lies in a *triple* intersection!

To remove all ambiguities, we need a whole tower of locally-defined differential forms in degree $p, p-1, \ldots, p-q, \ldots, 0$ defined on $1, 2, \ldots, q+1, \ldots, p+2$ -fold intersections.

The Čech-de Rham staircase...

We need all these pieces:

$$
\begin{array}{cc|cccccc}
\Lambda^{p+2} & & 0 & & & & & \\
\Lambda^{p+1} & \omega & \{\omega_\alpha\} & 0 & & & & \\
\Lambda^{p} & & \{A^p_\alpha\} & \delta\{A^p_\alpha\} = \{dA^{p-1}_{\alpha\beta}\} & & & & \\
\Lambda^{p-1} & \vdots & \vdots & \{A^{p-1}_{\alpha\beta}\} & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\Lambda^{1} & & & & \{A^1_{\alpha_0\ldots\alpha_{p-1}}\} & \delta\{A^1_{\alpha_0\ldots\alpha_{p-1}}\} & 0 & \\
\Lambda^{0} & & & & \ldots & \{A^0_{\alpha_0\ldots\alpha_p}\} & \delta\{A^0_{\alpha_0\ldots\alpha_p}\} & 0 \\
\hline
d & \uparrow & & & \ldots & & \{K_{\alpha_0\ldots\alpha_{p+1}}\} & \\
\delta & \rightarrow & \check{C}^0 & \check{C}^1 & \ldots & \check{C}^p & \check{C}^{p+1} & \check{C}^{p+2}
\end{array}
\; . \tag{8}
$$

Čech cohomology "consistency relations" then guarantees (almost) all the ambiguities vanish

The action for a WZ term

$$S[z] = \sum_{\alpha} \int_{c_{p,\alpha}} A^{p}_{\alpha} - \sum_{\alpha\beta} \int_{c_{(p-1),\alpha\beta}} A^{p-1}_{\alpha\beta} + \cdots + \\ +(-)^{p} \sum_{\alpha_0 \ldots \alpha_{p+1}} A^{0}_{\alpha_0 \ldots \alpha_p}(c_{0,\alpha_0 \ldots \alpha_p})$$

The quantization condition

What about the ambiguity in 0-forms, on $(p+2)$-fold intersections? Cannot be removed using forms in one lower degree...

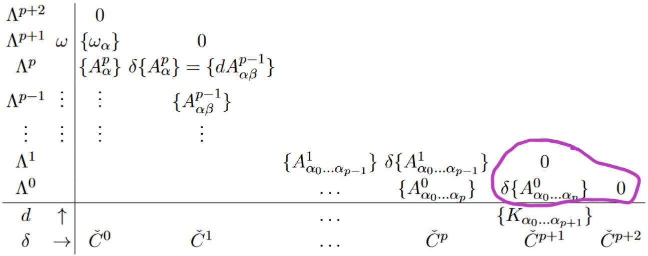
This 0-form ambiguity can be written

$$S' - S = K_{\alpha_0 \ldots \alpha_{p+1}},$$

where $K_{\alpha_0 \ldots \alpha_{p+1}}$ is an element of the Čech $(p+1)$-cochain $\{K\} := \delta\{A^0\}$

The quantization condition

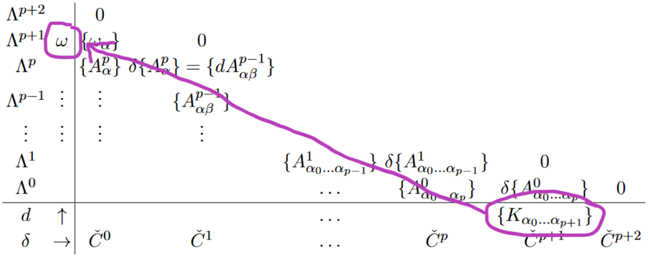
Recall from our staircase diagram that $\{K_{\alpha_0\ldots\alpha_{p+1}}\}$ is both d- and δ-closed:



d-closure means the 0-forms $K_{\alpha_0\ldots\alpha_{p+1}}$ are actually *constants*.

WZ action phase $e^{2\pi i S_{WZ}[z]}$ will be well-defined iff those constants are integers.

The quantization condition



δ-closure means $\{K_{\alpha_0\ldots\alpha_{p+1}}\}$ defines an integral cohomology class.
Going "back up the staircase" tells us:

$$[\omega] \in H^{p+1}(G/H, \mathbb{Z})$$

Some comments:

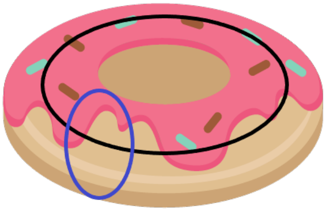
- If ω is in a trivial cohomology class, then coefficient of WZ term not quantized. *E.g.* $p = 1$ Landau levels on $\mathbb{R}^2$, $\omega = B dx \wedge dy$, $B \in \mathbb{R}$
- Can show that the action phase defined above reduces to $e^{2\pi i \int_b \omega}$ when evaluated on a boundary $z = \partial b$
- Can show that the action phase defined above is well-defined on $[\Sigma^p]$

The Manton condition for G-invariance of WZ terms

Defined using local forms, G-invariance of action phase is obscured - turns out it is *not* implied by G-invariance of ω!

Example: QM on the torus

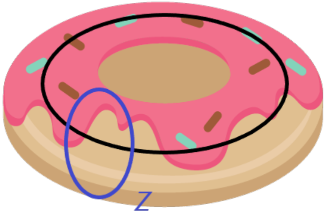
Homogeneous B-field defined by $U(1) \times U(1)$-invariant 2-form $F = \frac{B}{4\pi^2}\, d\theta_1 \wedge d\theta_2$



Cannot use Witten trick on non-trivial 1-cycles. Must integrate locally-defined forms.

Example: QM on the torus

E.g. on z, can integrate $A = -\frac{B}{4\pi^2}\theta_2 \, d\theta_1$



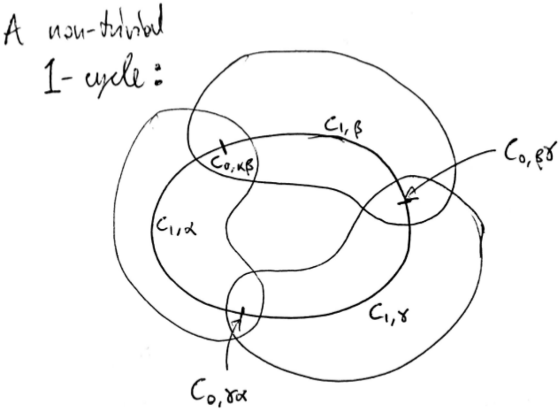
Action phase $e^{2\pi i S[z]} = e^{-iB\theta_2}$ not $U(1)$-invariant: only invariant for discrete shifts[6]

$$U(1) \times U(1) \to \mathbb{Z}_N \times \mathbb{Z}_N$$

[6] Manton, 1985

Now let's generalize this to any WZ term.

Condition for G-invariance: $p = 1$



$$S[z] = \int_{c_{1,\alpha}} A^1_\alpha - A^0_{\alpha\beta}(c_{0,\alpha\beta}) + \int_{c_{1,\beta}} A^1_\beta - A^0_{\beta\gamma}(c_{0,\beta\gamma}) + \int_{c_{1,\gamma}} A^1_\gamma - A^0_{\gamma\alpha}(c_{0,\gamma\alpha})$$

Condition for G-invariance: $p = 1$

$$S[z] = \int_{c_{1,\alpha}} A^1_\alpha - A^0_{\alpha\beta}(c_{0,\alpha\beta}) + \int_{c_{1,\beta}} A^1_\beta - A^0_{\beta\gamma}(c_{0,\beta\gamma}) + \int_{c_{1,\gamma}} A^1_\gamma - A^0_{\gamma\alpha}(c_{0,\gamma\alpha})$$

Compute variation under infinitesimal G-transformation generated by $X \in \mathfrak{g}$ using

$$L_X A^1_\alpha = \iota_X \omega_\alpha + d\iota_X A^1_\alpha, \quad \text{and} \quad L_X A^0_{\alpha\beta} = \iota_X dA^0_{\alpha\beta} = \iota_X (A^1_\alpha - A^1_\beta)$$

Stokes' theorem $\implies$

$$\delta_X S[z] = \int_{c_{1,\alpha}} \iota_X \omega_\alpha + \int_{c_{1,\beta}} \iota_X \omega_\beta + \int_{c_{1,\gamma}} \iota_X \omega_\gamma = \int_z \iota_X \omega$$

Generalizes to any cycle, in general p.

$$\delta_X S[z] = \sum \int_\alpha L_X A^p_\alpha - \sum \int_{\alpha\beta} L_X A^{p-1}_{\alpha\beta} + \cdots = \int_z \iota_X \omega$$

Demand $\delta_X S[z] = 0$ on all cycles $\overset{\text{dR}}{\Longrightarrow}$

$$\boxed{\iota_X \omega = df_X, \quad f_X \in \Lambda^{p-1}(G/H)} \tag{9}$$

$=$ the 'Manton condition'. Nec and suff when G connected.

A broad generalization of the 'anomaly' Manton observed for QM on T^2 to any homogeneous space sigma model in QFT

Noether currents for G-invariance

Noether currents = the $(p-1)$-forms f_X that appear in the Manton condition $\iota_X \omega = df_X$

If the Manton condition fails, the f_X, and hence the Noether currents, are not globally-defined

Summary of classification

1. AB terms, classified by

$$H^p_{dR}(G/H,\mathbb{R})/H^p(G/H,\mathbb{Z})_{\mathbb{R}}$$

if neglecting torsion

2. WZ terms, classified by

$$\{\omega \in Z^{p+1}(G/H,\mathbb{Z}) \mid \iota_X(\omega) = df_X \ \forall X \in \mathfrak{g}\}$$

(Evidence for this classification from differential cohomology of G/H - see backup slides)

Applications: the Composite Higgs

Example cosets

Require pNGBs $\supset (\mathbf{2},\mathbf{2})$ under $SU(2)_L \times SU(2)_R$. Leaves many viable cosets.

G	H	N_G	Reps	AB terms	WZ terms
$SO(5)$	$SO(4)$	4	$(\mathbf{2},\mathbf{2})$	$U(1)$	-
$SO(6)$	$SO(5)$	5	$(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{1})$	-	$\mathbb{Z}$
$SO(5)\times U(1)$	$SO(4)$	5	$(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{1})$	$U(1)$	-
$SO(6)$	$SO(4)\times SO(2)$	8	$2\times(\mathbf{2},\mathbf{2})$	$U(1)\times U(1)$	-
$SO(6)$	$SO(4)$	9	$2\times(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{1})$	$U(1)$	$\mathrm{Z}\times\mathbb{R}^4$
$SU(5)$	$SO(5)$	14	$(\mathbf{3},\mathbf{3})+(\mathbf{2},\mathbf{2})+(\mathbf{1},\mathbf{1})$	-	$\mathbb{Z}$

The $SO(5)/SO(4) \simeq S^4$ Model[7]

AB term

$$S[z] = \frac{\theta}{V_{S^4}} \int_z d^4 H = \theta W, \quad \theta \in U(1)$$

Physical effects (if any) non-perturbative, expect become important in UV (from instanton argument)

[7] Agashe, Contino, Pomarol, 2005

The $SO(6)/SO(5) \simeq S^5$ model[8]

WZ term, can write using Witten construction since $H_4(S^5) = 0$:

$$S[z = \partial B] = \frac{n_{WZ}}{V_5} \int_B d\eta \wedge d^4 H, \quad n \in \mathbb{Z}$$

Physics? Dimension-9, but gauging $SU(2)_L$ produces dimension-5 operators.

[8] Gripaios, Pomarol, Riva, Serra, 2009

Probing the microscopic theory

Has been suggested the $SO(6)/SO(5)$ model can arise from an $Sp(2N_c)$ gauge theory with 4 fundamental Weyl fermions.[9] Global symmetry breaking

$$SU(4) \simeq SO(6) \to Sp(4) \simeq SO(5)$$

Anomaly-matching would then predict

$$n_{WZ} = 0$$

So measuring WZ-induced processes probes gauge group of UV completion

[9] Barnard, Gherghetta, Ray, 2014

The $SO(5) \times U(1)/SO(4) \simeq S^4 \times S^1$ model[10]

Local coordinates $(h_1, h_2, h_3, h_4, \eta) = (H, \eta)$. WZ term from $SO(5) \times U(1)$ volume form $\omega = d\eta\ d^4H$? Would induce *e.g.* $\eta \to WWZh$ rare decay

No!

Manton condition violated for generator of $U(1)$:

$$\iota_{\partial_\eta}\omega = \mathrm{Vol}_{S^4}$$

closed but not exact. Same physics as QM on T^2.

[10] Gripaios, You, Nardecchia, 2015

The $SO(5) \times U(1)/SO(4) \simeq S^4 \times S^1$ model

We can see this explicitly.

$H_4(S^4 \times S^1, \mathbb{Z}) = \mathbb{Z}$; need to use local forms *E.g.*

$$S[z] = \int_z \eta_0 \; d^4 H = \eta_0 V_{S^4},$$

not $U(1)$-invariant! $U(1)$ is broken to discrete subgroup

The $SO(6)/SO(4)$ model

As a CHM:

- features 9 pNGBs: 2 HDs, 1 singlet

$$\phi_a \hat{T}^a = \begin{pmatrix} \mathbf{0}_{4\times 4} & H_A^T & H_B^T \\ -H_A & 0 & \eta \\ -H_B & -\eta & 0 \end{pmatrix}$$

- $SO(6)/SO(4) \simeq SU(4)/SO(4)$, imagine UV completion as an $SO(N_c)$ gauge theory w 4 fund Weyl fermions

The $SO(6)/SO(4)$ model

Topologically, the Stiefel manifold $SO(6)/SO(4)$ is an S^4 bundle over S^5 (unit tangent bundle of S^5)

Cannot use Witten construction because

$$H_4(E, \mathbb{Z}) = \mathbb{Z}$$

Computing space of WZ terms

Need to compute space of $SO(6)$-invariant, integral, closed 5-forms on $SO(6)/SO(4)$, that satisfy Manton condition

Lemma: Manton condition guaranteed when G is semi-simple

Proof: Semi-simple means $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, i.e. for any $X \in \mathfrak{g}$, $X = [Y, Z]$. Using $[L_Y, \iota_Z]\alpha = \iota_{[Y,Z]}\alpha$,

$$\iota_X \omega = \iota_{[Y,Z]}\omega = d(\iota_Y \iota_Z \omega), \tag{10}$$

which is Manton condition $\forall X \in \mathfrak{g}$

$SO(6)$ is semi-simple

Computing space of WZ terms

Need to compute space of $SO(6)$-invariant, integral, closed 5-forms on $SO(6)/SO(4)$, that satisfy Manton condition

1. $SO(6)$ semi-simple, $L_X\omega = 0 \implies$ Manton condition. Hence reduces to space of integral cocycles in relative Chevalley-Eilenberg cohomology
2. $G = SO(6)$ is compact, connected, and $H = SO(4)$ is connected. Hence reduces to space of integral cocycles in relative Lie algebra cohomology[11]

So the computation reduces to algebra!

[11] Chevalley, Eilenberg, 1948

Computing space of WZ terms

Fortunately, there is a package in `Maple`.

We find the following basis for WZ terms:

$$\{d^4 H_B d\eta,\ d^4 H_A d\eta,\ \epsilon_{ijkl} dh^i_A dh^j_B dh^k_B dh^l_B d\eta,$$
$$\epsilon_{ijkl} dh^i_A dh^j_A dh^k_B dh^l_B d\eta,\ \epsilon_{ijkl} dh^i_A dh^j_A dh^k_A dh^l_B d\eta\},$$

Space of WZ terms is thus $\mathbb{Z} \times \mathbb{R}^4$

Space of AB terms turns out to be $U(1)$

Summary

- Classified topological terms in G/H sigma model starting from singular homology
- AB terms and WZ terms
- Manton condition for G-invariance of WZ terms
- Composite Higgs examples

Outlook

hep-th

Differential characters.

Beyond differential characters. e.g. in theory
$\phi : \Sigma^4 \simeq S^4 \to SU(2) \simeq S^3$, topological term because
$\pi_4(S^3) = \mathbb{Z}_2 \to$ fermionic solitons

hep-ph

Explore Composite Higgs pheno. Requires we gauge the topological terms.

Thanks!

Backup: Differential Characters etc

The geometry of WZ terms

$p = 1$

The Čech data define a principal $U(1)$ bundle over G/H

- $\{A^1_\alpha\}$ is connection/ 'background gauge field'
- ω is curvature/ 'field strength'
- Quantization condition corresponds to integrality of c_1
- $S_{WZ}[z]$ is holonomy for z

$p = 2$

The Čech data define a principal Hitchin gerbe over G/H

- $\{A^2_\alpha\}$ is 2-form connection
- ω is 3-form curvature
- $S_{WZ}[z]$ is "higher holonomy" for z

Evidence for classification: differential cohomology

A topological term (as we have defined it) is a differential character.[12]

[12] Cheeger-Simons, 1985.

Evidence for classification: differential cohomology

Definition: A differential character f is a homomorphism from smooth singular cycles to $U(1)$,

$$f : Z_p(G/H, \mathbb{Z}) \to U(1),$$

such that for every $(p+1)$-chain c,

$$f(z = \partial c) = e^{2\pi i \int_c \omega},$$

where $(p+1)$-form ω is *curvature* of the character f (uniquely determined)

Evidence for classification: differential cohomology

Space of differential characters forms an abelian group,

$$\hat{H}^p(G/H,\mathbb{Z}) := \{f \in \mathrm{Hom}(Z_p(G/H,\mathbb{Z}), U(1)) \mid f(\partial c) = e^{2\pi i \int_c \omega}\}, \tag{11}$$

which sits inside an exact sequence:

$$0 \to H^p(G/H,U(1)) \to \hat{H}^p(G/H,\mathbb{Z}) \to \Omega_0^{p+1} \to 0. \tag{12}$$

if no torsion, sequence splits, and group of characters is direct product of two groups: AB and WZ (need to figure out how to impose G-invariance).

Low-degree differential cohomology groups

- $\hat{H}^0(M, \mathbb{Z}) = C^\infty(M, U(1))$
- $\hat{H}^1(M, \mathbb{Z})$ is space of $U(1)$ bundles over M with connection
- $\hat{H}^2(M, \mathbb{Z})$ is space of Hitchen gerbes over M with connection[13]

[13] Hitchin, 2001

Classifying AB terms II: locally-defined p-form

If allow A to be only locally-defined on open sets, turns out space of AB terms classified by

$$H^p(G/H, U(1)),$$

pth singular cohomology of G/H valued in $U(1)$. Includes torsion...